(1) Linear differential equation

Def:- A differential eqn. of the form $\frac{dy}{dx} + Py = Q$ where P and Q are constant or function of x only, is known as first order linear differential equation

its I.F. $= e^{\int P dx}$

Working rule :- To solve diff. eqn. first find I.F. $= e^{\int P dx}$ Then multiplying the given linear eqn. by I.F. and integrate then

L.H.S. $= y \times$ I.F. and R.H.S $=$
$\int Q \times \text{I.F} \, dx + C$

Problem (1) Solve $\frac{dy}{dx} + \frac{1}{x}\sin 2y = x^3 \cos^2 y$

Soln:- $\sec^2 y \frac{dy}{dx} + \frac{2\sin y \cos y}{x \cos^2 y} = x^3$

$\Rightarrow \sec^2 \frac{dy}{dx} + 2\frac{\sin y}{\cos y} \cdot \frac{1}{x} = x^3$

We put $\tan y = t$, $\sec^2 y \frac{dy}{dx} = \frac{dt}{dx}$

$\Rightarrow \frac{dt}{dx} + 2t\frac{1}{x} = x^3$ —— (1)

I.F. $= e^{\int \frac{2}{x} dx} = e^{2\log x}$
$= e^{\log x^2} = x^2$

Multiplying (1) by I.F. we get

$$t \cdot x^2 = \int x^3 x^2\, dx$$

$$x^2 \tan y = \frac{x^6}{6} + K$$

$$\therefore 6x^2 \tan y = x^6 + c \text{ where } 6K = c$$

<u>problem</u> (2) $\frac{dy}{dx} + \frac{xy}{1-x^2} = x\sqrt{y}$

soln:— we put $\sqrt{y} = t$

$$\therefore \frac{1}{2\sqrt{y}} \frac{dy}{dx} = \frac{dt}{dx}$$

$$\therefore \frac{1}{\sqrt{y}} \frac{dy}{dx} = \frac{y}{\sqrt{y}} \frac{x}{1-x^2} = x$$

$$\Rightarrow \frac{dt}{dx} + \frac{x}{2(1-x^2)} t = \frac{x}{2} \quad \text{—— (1)}$$

$$\therefore \text{I.F.} = e^{\frac{1}{2}\int \frac{x}{1-x^2} dx}$$

$$= e^{-\frac{1}{4}\int \frac{-2x}{1-x^2} dx}$$

$$= e^{-\frac{1}{4}\log(1-x^2)}$$

$$= e^{\log(1-x^2)^{-\frac{1}{4}}} = (1-x^2)^{-\frac{1}{4}}$$

Multiplying (1) by I.F. and integrate we get

$$t \cdot (1-x^2)^{-\frac{1}{4}} = \int \frac{x}{2} (1-x^2)^{-\frac{1}{4}} dx$$

$$(1-x^2)^{-\frac{1}{4}} \sqrt{y} = -\frac{1}{4} \int \frac{-2x}{(1-x^2)^{1/4}} dx$$

we put $1-x^2 = t$

$$-2x\, dx = dt$$

$$= -\frac{1}{4} \int \frac{dt}{t^{1/4}}$$

$$-\frac{1}{4}\frac{t^{-\frac{1}{4}+1}}{-\frac{1}{4}+1}+K$$

$$(1-x^2)^{-\frac{1}{4}}\sqrt{y} = -\frac{1}{4}\frac{t^{3/4}}{3/4}+K$$

$$= -\frac{1}{3}(1-x^2)^{3/4}+K$$

$$\Rightarrow (1-x^2)^{-\frac{1}{4}}\sqrt{y}+\frac{1}{3}(1-x^2)^{3/4} = K$$

Problem (3) Solve $\sec x \frac{dy}{dx} - y = \sin x$

Solution:- $\frac{dy}{dx} - \cos x \cdot y = \frac{\sin x}{\sec x}$

$$\Rightarrow \frac{dy}{dx} - \cos x \cdot y = \sin x \cos x \quad \text{--- (1)}$$

$$I.F. = e^{-\int \cos x\, dx} = e^{-\sin x}$$

Multiplying (1) I.F and integrate we get

$$y \times I.F. = \int \sin x \cdot \cos x \cdot e^{-\sin x} dx$$

we put $\sin x = t$, $\cos x\, dx = dt$

$$\Rightarrow y \times e^{-\sin x} = \int t e^{-t} dt$$

$$\Rightarrow y \times e^{-\sin x} = -(t+1)e^{-t}+K$$

$$\Rightarrow y e^{-\sin x} = -(\sin x+1)e^{-\sin x}+K$$

$$y = -(\sin x+1)+K e^{\sin x}$$

$$\therefore y+\sin x+1 = K e^{\sin x}$$

Problem (4) $\frac{dy}{dx}+\frac{y}{x} = y^2$

Solution:- $\frac{1}{y^2}\frac{dy}{dx}+\frac{1}{y}\cdot\frac{1}{x}$

Soln:- $\frac{1}{y^2}\frac{dy}{dx}+\frac{1}{y}\frac{1}{x}=1$

we put $\frac{1}{y}=t$

$$\frac{1}{y^2}\frac{dy}{dx}=-\frac{dt}{dx}$$

$$\Rightarrow -\frac{dt}{dx}+t\cdot\frac{1}{x}=1$$

$$\Rightarrow \frac{dt}{dx}-t\cdot\frac{1}{x}=-1 \quad \text{— (1)}$$

$$I.F.=e^{-\int\frac{1}{x}dx}=e^{-\log x}=e^{\log\frac{1}{x}}=\frac{1}{x}$$

Multiplying (1) by I.F. and integrating we get

$$t\times I.F.=\int(-1)\frac{1}{x}dx$$

$$\Rightarrow \frac{1}{xy}=-\log x+K$$

$$\Rightarrow \frac{1}{xy}=\log\frac{1}{x}+K$$

Problem (4) $(x^3y^2+xy)dx=dy$

Soln:- $\frac{dy}{dx}-xy=x^3y^3$ | we put $-\frac{1}{y}=t$, $\frac{1}{y^2}\frac{dy}{dx}=\frac{dt}{dx}$

$$\Rightarrow \frac{1}{y^2}\frac{dy}{dx}-\frac{1}{y}x=x^3$$

$$\therefore \frac{dt}{dx}+tx=x^3$$

$$I.F.=e^{\int x\,dx}=e^{x^2/2}$$

$$\therefore t\cdot e^{x^2/2}=\int x^3\cdot e^{x^2/2}dx$$

we put $\frac{x^2}{2}=p$ $\therefore xdx=dp$ and $x^2=2p$

$$\Rightarrow -\frac{1}{y}e^{x^2/2}=\int 2p\,e^p\,dp$$

$$=2(p-1)e^p+K \quad |\text{by parts}$$

$$\therefore -\frac{e^{x^2/2}}{y}=2\left(\frac{x^2}{2}-1\right)e^{x^2/2}+K$$

— x —